Anhang A

Lösungen zu den Übungsaufgaben

A

A

A Lösungen zu den Übungsaufgaben

A.1 Lösungen zu Zahlen, Gleichungen und Gleichungssystemen

1.1 a) $\{2,3,5,7,11,13,17,19\}$ b) $\emptyset$

1.2 (i) $A \cap B = \{x : 1 \le x < 2\}$,
(ii) $A \cup B = \{x : 0 < x \le 3\}$,
(iii) $A \times B = \{(x,y) : 0 < x < 2 \text{ und } 1 \le y \le 3\}$,
(iv) $A \backslash B = \{x : 0 < x < 1\}$.

1.3 a) $M_1 \cup M_2 = \{2,3,4,6,8,9,10,12,14,15,16,18,\ldots\}$
$M_1 \cap M_2 = \{6,12,18,\ldots\}$, $M_1 \backslash M_2 = \{2,4,8,10,14,16,\ldots\}$
$M_2 \backslash M_1 = \{3,9,15,21,\ldots\}$
b) $M_1 = \{1,-2\}$, $M_2 = \{1,2\}$, $M_1 \cap M_2 = \{1\}$, $M_1 \cup M_2 = \{1,2,-1,-2\}$,
$M_1 \backslash M_2 = \{-2\}$, $M_2 \backslash M_1 = \{2\}$

1.7 a) 1, 1, 3, 3, 1, 4, 6, 4, 1 b) 108 243 216

1.9 $\binom{n}{k} \frac{1}{n^k} = \frac{n!}{(n-k)!\,k!}\,\frac{1}{n^k} = \frac{1}{k!}\,\frac{1\cdot 2 \cdot \ldots \cdot (n-k) \qquad (n-k+1)\cdot\ldots\cdot(n)}{\underbrace{1\cdot 2\cdot\ldots\cdot(n-k)}_{(n-k)\ Zahlen} \cdot \underbrace{n^k}_{k\ Zahlen}} \le \frac{1}{k!}\cdot 1$

1.10 $x^5 + 20\,x^4 + 160\,x^3 + 640\,x^2 + 1280\,x + 1024$
$625\,y^4 - 500\,y^3 + 150\,y^2 - 20\,y + 1$
$a^6 - 6\,a^4\,b + 12\,a^2\,b^2 - 8\,b^3$

1.11 sum(kˆ2+1, k=71..125);

1.12 sum(kˆ3, k=1..n)=normal(sum(kˆ3, k=1..n));

1.13 a) $(\frac{9}{2})^2 x^{4a-3} y$ b) $a^3 + a^2 b + ab^2$

1.14 a) $\frac{2x(2x^2-3r^2)}{\sqrt{r^2-x^2}}$ b) $k^2/\sqrt{(x-k)^2+x^2}$ c) $3(x-1)$

1.15 a) ab^2 b) $a^{2/3}$ c) $ab^{4/3}$ d) $a^{13/8}$ e) $a^{15/32}$

1.16 a) $1/2,\ 3$ b) $3/2,\ -1/3,\ 12$ c) $\frac{n}{n+1}\log a - \frac{1}{m(n+1)}\log b$

1.18 a) $\mathbb{L} = \{-5,\ 3\}$ b) $\mathbb{L} = \emptyset$ c) $\mathbb{L} = \left\{\frac{5}{3},\ \frac{7}{3}\right\}$ d) $\mathbb{L} = \{-2\}$ e) $\mathbb{L} = \{-1\}$

1.19 $c = -2$

1.20 a) $\mathbb{L} = \{0,\ 2\}$ b) $\mathbb{L} = \{\pm 2,\ \pm 3\}$ c) $\mathbb{L} = \left\{-3,\ \pm\sqrt{2},\ \pm 5\right\}$

1.21 a) $\mathbb{L} = \{3.5\}$ b) $\mathbb{L} = \emptyset$ c) $\mathbb{L} = \emptyset$ d) $\mathbb{L} = \{-1\}$

1.22 a) $\mathbb{L} = \{-4.424,\ 5.424\}$ b) $\mathbb{L} = \{-2,\ 1\}$

1.23 a) $\mathbb{L} = (8,\ \infty)$ b) $\mathbb{L} = \mathbb{R}$ c) $\mathbb{L} = \emptyset$ d) $\mathbb{L} = (-2.562,\ 1.562)$

1.24 a) $x_1 = x_2 = x_3 = 1$
b) $x_1 = 1,\ x_2 = 3,\ x_3 = 2$
c) $\mathbb{L} = \emptyset$

1.25 a) $\mathbb{L} = \left\{ (x_1,\ x_2,\ x_3) \in \mathbb{R}^3 : x = \begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} \right\}$

b) $\mathbb{L} = \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R} \right\}$

c) $\mathbb{L} = \emptyset$

1.26 a) $\mathbb{L} = \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} ;\ \lambda,\ \mu \in \mathbb{R} \right\}$

b) $\mathbb{L} = \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} ;\ \lambda,\ \mu \in \mathbb{R} \right\}$

c) $\mathbb{L} = \emptyset$

1.27 Die homogenen Systeme sind immer lösbar.

A.2

A.2 Lösungen zur Vektorrechnung

2.1 a) $\vec{s}_1 = \begin{pmatrix} 1 \\ 20 \\ -18 \end{pmatrix}$; $|\vec{s}_1| = 26.92$ b) $\vec{s}_2 = \begin{pmatrix} 5 \\ -24 \\ 2 \end{pmatrix}$; $|\vec{s}_2| = 24.59$

c) $\vec{s}_3 = \begin{pmatrix} -19 \\ 36 \\ -22 \end{pmatrix}$; $|\vec{s}_3| = 46.27$ d) $\vec{s}_4 = \begin{pmatrix} 170 \\ -60 \\ -40 \end{pmatrix}$; $|\vec{s}_4| = 184.66$

2.2 $F = -(F_1 + F_2 + F_3 + F_4) = \begin{pmatrix} -200 \\ -175 \\ 10 \end{pmatrix} N$

2.3 $\vec{e}_a = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$; $\vec{e}_b = \frac{1}{\sqrt{38}} \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$; $\vec{e}_c = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

2.4 $\vec{e} = -\frac{\vec{a}}{|\vec{a}|} = \frac{1}{5} \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$

2.5 $\vec{r}\,(Q) = \vec{r}\,(P) + 10\,\frac{\vec{a}}{|\vec{a}|} = \begin{pmatrix} -7.16 \\ -6.08 \\ -1.08 \end{pmatrix}$

2.6 $\vec{r}\,(Q) = \vec{r}\,(P_1) + \frac{1}{2}\,\overrightarrow{P_1 P_2} = \begin{pmatrix} 0.5 \\ 3.5 \\ 2.5 \end{pmatrix}$

2.7 a) 4 b) 96 c) 22

2.8 a) $\varphi = 48.47°$ b) $\varphi = 156.5°$

2.10 $\vec{c} = \vec{a} + \vec{b}$; $\vec{a} \cdot \vec{b} = 0$

2.11 a) $|\vec{a}| = \sqrt{3}$, $\alpha = \beta = \gamma = 54,74°$

b) $|\vec{a}| = \sqrt{30}$, $\alpha = 24.09°$, $\beta = 111.42°$, $\gamma = 79.48°$

2.12 $|\vec{a}| = \overline{BC} = 2\sqrt{6}$ $\left|\vec{b}\right| = \overline{AC} = 2\sqrt{14}$ $|\vec{c}| = \overline{AB} = 2\sqrt{14}$

$\alpha = 38.21°$ $\beta = 70.89°$ $\gamma = 70.89°$

2.13 $\vec{b}_a = \frac{1}{3} \begin{pmatrix} 22 \\ -22 \\ 11 \end{pmatrix}$ $\vec{b}_a = \frac{1}{9} \begin{pmatrix} -28 \\ 28 \\ -14 \end{pmatrix}$

2.14 Es ist $\gamma = 90°$, $a_x = 8.66$, $a_y = 5$, $a_z = 0$.

2.15 a) $\alpha = 103.6°$ $\beta = 76.37°$ $\gamma = 19.47°$
b) $\alpha = 42.03°$ $\beta = 68.19°$ $\gamma = 123.9°$

2.16 a) $\begin{pmatrix} 3 \\ -10 \\ 8 \end{pmatrix}$ b) $\begin{pmatrix} -12 \\ -18 \\ -3 \end{pmatrix}$ c) $\begin{pmatrix} 15 \\ 8 \\ -18 \end{pmatrix}$ d) $\begin{pmatrix} -10 \\ 14 \\ 12 \end{pmatrix}$

2.17 $\overrightarrow{F}_R = \sum_{i=1}^{4} F_i = \begin{pmatrix} 167.55 \\ -148.68 \end{pmatrix}$ $\left|\overrightarrow{F}_R\right| = 224\,N,$ $\alpha = 41.6°$

2.18 a) $|F| = 30\,N$ $|a| = 3$ b) $\varphi = 63.61°$
c) $\overrightarrow{F}_a = 4.444 \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ $\left|\overrightarrow{F}_a\right| = 13.33$ d) $\overrightarrow{a} \cdot \overrightarrow{b} = 0$

2.19 a) $\varphi = 60°$ b) $\overrightarrow{M} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix} Nm;$ $\left|\overrightarrow{M}\right| = 5.2\,Nm$ c) $\overrightarrow{F}_r = \frac{1}{2} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} N$

2.20 $\overrightarrow{F} \cdot \overrightarrow{s} = 4\,Nm$ $\overrightarrow{F}\,\overrightarrow{S}_1 + \overrightarrow{F}\,\overrightarrow{S}_2 = 4\,Nm \Rightarrow$ Die Arbeit ist wegunabhängig.

2.22 $g : \overrightarrow{x} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix};$ $\lambda = 1 : Q_1 = (3,\, 0,\, 2)$; $\lambda = 2 : Q_2 = (2,\, 0,\, 1)$; $\lambda = -5 : Q_3 = (9,\, 0,\, 8)$

2.23 $g : \overrightarrow{x} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix}$

2.24 Ja: $\overrightarrow{x} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$; $P_3 : \lambda = 2$

2.25 $d = 1,22$

2.26 $g : \overrightarrow{x} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \sqrt{\frac{3}{2}} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$

2.27 a) g_1 und g_2 sind windschief; $d = 2.04$.
b) Geraden sind parallel, da $\overrightarrow{a} \| \overrightarrow{b}$; $d = 1.79$
c) Geraden schneiden sich genau in einem Punkt $S = (5,\, 2,\, 10)$; $\alpha = 32.4°$

2.28 g_1 und g_2 sind windschief zueinander; $d = 2.85$.

2.29 $E = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$; $\overrightarrow{n} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$; $Q = (10,\, 9, 11)$

2.30 $\overrightarrow{r}(P) = \overrightarrow{r}_1 + \lambda\,(\overrightarrow{r}_2 - \overrightarrow{r}_1) + \mu\,(\overrightarrow{r}_3 - \overrightarrow{r}_1) = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix}$

2.31 Ja: $E = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \\ 12 \end{pmatrix} \Rightarrow \lambda = 1\,,\ \mu = 3.$

2.32 $4x + 3y + z = 54$

2.33 a) g und E schneiden sich, da $\overrightarrow{n} \cdot \overrightarrow{a} = 2 \neq 0$. Schnittpunkt $\lambda_s = 4.5$
$\Rightarrow S = (18.5,\, 5.5,\, 11)$. Schnittwinkel $\varphi = 9.27°$
b) $g \| E$, da $\overrightarrow{n} \cdot \overrightarrow{a} = 0$; Abstand $d = 1.51$

c) $E = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$; $\overrightarrow{n} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$, $g = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \\ 15 \end{pmatrix}$.

$\Rightarrow$ Schnittpunkt $S = (1, -2, -2)$; Schnittwinkel $\varphi = -22.79°$

2.34 $E_1 \| E_2$, da $n_1 \times n_2 = \overrightarrow{0}$; Abstand $d = 3.74$

2.35 $E_1 \nparallel E_2$, Schnittgerade $\overrightarrow{r} = \frac{1}{3} \begin{pmatrix} 0 \\ 59 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix}$, Schnittwinkel $\varphi = 27.2°$

2.36 Nein: $\overrightarrow{a}_3 = \overrightarrow{a}_1 + \overrightarrow{a}_2$: die Vektoren sind linear abhängig.

2.37 Ja.

2.38 $\overrightarrow{b} = \overrightarrow{a}_1 + \overrightarrow{a}_2 - 2\overrightarrow{a}_3 - \overrightarrow{a}_4$.

2.39 Linear abhängig, da $\det(\overrightarrow{a}_1, \overrightarrow{a}_2, \overrightarrow{a}_3, \overrightarrow{a}_4, \overrightarrow{a}_5) = 0$.

2.40 a) $\overrightarrow{b} = \overrightarrow{a}_1 + \overrightarrow{a}_2 + \overrightarrow{a}_3$. b) Nein.

2.41 $\overrightarrow{d} = -2\overrightarrow{a}_1 + \overrightarrow{a}_2 - \overrightarrow{a}_3$.

A.3 Lösungen zu Matrizen und Determinanten

3.1 a) $\begin{pmatrix} 6 & 32 \\ 13 & 13 \\ -8 & 7 \end{pmatrix}$ b) $\begin{pmatrix} -7 & -3 \\ 1 & -4 \\ 0 & 4 \end{pmatrix}$ c) $\begin{pmatrix} 29 & 23 & 10 \\ 11 & 18 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 8 & 13 \\ 8 & 8 \\ -1 & 2 \end{pmatrix}$

e) $\begin{pmatrix} -2 & 4 & -3 \\ 9 & 2 & 4 \end{pmatrix}$ f) $\begin{pmatrix} -2 & 9 \\ 4 & 2 \\ -3 & 4 \end{pmatrix}$

3.2 a) $A^2 = \begin{pmatrix} 4 & 18 & 3 \\ 0 & 16 & 5 \\ 0 & 0 & 1 \end{pmatrix}$, $B^2 = \begin{pmatrix} 3 & -6 & 2 \\ -4 & 7 & -4 \\ 4 & -6 & 5 \end{pmatrix}$,

$A \cdot B = \begin{pmatrix} -1 & 6 & 1 \\ -3 & 5 & -4 \\ 1 & -3 & 0 \end{pmatrix}$, $B \cdot A = \begin{pmatrix} 2 & 3 & 2 \\ -2 & 5 & 1 \\ 2 & -9 & -3 \end{pmatrix}$

b) $A \cdot B = \begin{pmatrix} 1 & 26 \\ 0 & 6 \end{pmatrix}$ $B \cdot A = \begin{pmatrix} 0 & 2 & 0 & 2 \\ -2 & -1 & 0 & -2 \\ 10 & 18 & 0 & 23 \\ 2 & 7 & 0 & 8 \end{pmatrix}$

3.3 a) $A^{-1} = \frac{1}{9} \begin{pmatrix} -1 & -1 & 5 \\ 3 & 3 & -6 \\ 2 & 11 & -10 \end{pmatrix}$ b) $B^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & -4 \end{pmatrix}$

c) $C^{-1} = \frac{1}{3} \begin{pmatrix} -2 & 1 & -3 & -3 \\ -3 & 3 & -3 & -6 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 3 \end{pmatrix}$

3.5 $A = \begin{pmatrix} 3 & 2 & -1 & 0 \\ 5 & 1 & 2 & 0 \\ 4 & 5 & 1 & 0 \end{pmatrix}$, $(18, 22, 38)$.

3.6 5, 0, λ

3.8 -12, -21, -53

3.9 a) $0\,,\quad -1$ b) $1\,,\quad 2\,,\quad 3$

3.10 a) 142 b) 180

3.11 $\det A = -8\,,\quad (x_1,\,x_2,\,x_3) = (-3,\,3,\,0)$

3.12 $A^{-1} = \frac{1}{9}\begin{pmatrix} -1 & -1 & 5 \\ 3 & 3 & -6 \\ 2 & 11 & -10 \end{pmatrix} \quad B^{-1} = \frac{1}{7}\begin{pmatrix} 3 & -2 & -2 \\ 0 & 0 & 7 \\ -1 & 3 & -11 \end{pmatrix}$

3.13 $Rang\,(A) = 3 \quad Rang\,(B) = 3 \quad Rang\,(C) = 3 \quad Rang(D) = 3.$

3.14 $\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} = 2 \neq 0 \quad \vec{x} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

3.15 a) $Rang\,(A) = 2 \quad Rang\,(A/b) = 3 \Rightarrow$ nicht lösbar.
b) $Rang\,(A) = 2 = Rang\,(A/b) \Rightarrow$ lösbar, nicht eindeutig
z.B. $(-2,\,1,\,-1)$ ist Lösung.

3.16 a) $\det\left(\vec{a},\,\vec{b},\,\vec{c}\right) = 0 \Rightarrow$ linear abhängig
b) $\det\left(\vec{a},\,\vec{b},\,\vec{c}\right) \neq 0 \Rightarrow$ linear unabhängig, $\vec{d} = -3\vec{a} + \vec{b} + 2\vec{c}$.

A.4 Lösungen zu elementaren Funktionen

A.4

4.1 a) $\mathbb{D} = \{x : |x| \geq 1\} \quad \mathbb{W} = \mathbb{R}_{\geq 0}$
b) $\mathbb{D} = \mathbb{R} \setminus \{0\} \quad \mathbb{W} = \mathbb{R}$
c) $\mathbb{D} = \mathbb{R} \setminus \{-2,\,2\} \quad \mathbb{W} = (-\infty,\,0] \cup \left(\frac{1}{4},\,\infty\right)$
d) $\mathbb{D} = \mathbb{R} \setminus \{-1\} \quad \mathbb{W} = \mathbb{R} \setminus \{1\}$
e) $\mathbb{D} = \mathbb{R} \quad \mathbb{W} = \mathbb{R}_{\geq 1}$
f) $\mathbb{D} = \mathbb{R} \quad \mathbb{W} = \left[-\frac{1}{2},\,+\frac{1}{2}\right]$

4.2 a) gerade b) ungerade c) ungerade d) gerade e) gerade f) -

4.3 a) streng monoton fallend in $\mathbb{R}_{\leq 0}$; streng monoton wachsend in $\mathbb{R}_{\geq 0}$
b) streng monoton wachsend
c) streng monoton wachsend
e) streng monoton wachsend

4.4 a) $y = \frac{1}{2\,x} \quad \mathbb{D} = \mathbb{R}_{>0}$
b) $y = \frac{1}{3}\,x^2 \quad \mathbb{D} = \mathbb{R}_{\geq 0}$
c) $y = \ln x + 0,5 - \ln 2 \quad \mathbb{D} = \mathbb{R}_{>0}$
d) $y = -\frac{x+1}{x-1} \quad \mathbb{D} = (-\infty,\,1)$

4.5 $y = -2\,x + 5$

4.6 a) $1 \quad 2 \quad -5$ b) -1 c) $0 \quad 2 \quad 5 \quad -5$

4.7 a) $f\,(2) = -5$ b) $f\,(3) = 49.1$

4.8 Ja: z.B. $x^2 + 1$

4.9 $y = x^3 - 2\,x + 1$

4.10 a) -1 (doppelt), 1 b) $\pm 2,\,\pm 3$

4.11 $f\,(x) = \frac{1}{2}\,x^3 + \frac{1}{2}\,x + 1$

4.12 factor(",x)

4.13 fsolve(",x)

4.14 unapply

4.15 plot

4.16 factor, convert(”, ’horner’), degree

4.17 a) $NS: -2, 1$ $Pol: 2$ b) $NS: 3, 4$ $Pol: -1, 0$ c) $NS: 1$ $Pol: -1$ d) $NS: -$ $Pol: \pm 1$

4.18 a) $NS: \pm 2$ $Pol: -$ $Asymptote: y = 1$ b) $NS: 2$ *doppelt* $Pol: -2$ $Asymptote: x - 6$
c) $NS: 1$ $Pol: 2$ $Asymptote: y = 1$ d) $NS: 1$ *doppelt* $Pol: -1$ *doppelt* $Asymptote: y = 1$

4.19 plot, numer, denom, factor, normal, asympt, solve

4.21 $t = 2.3\,RC$

4.22 $t = 1.5\,s$

4.23 $a = 8 \quad b = 0.4159$

4.24 a) $x_1 = -0.3012$ $x_2 = \ 2.3012$ b) Subtitution $t = e^x$. $x_1 = 0$, $x_2 = 0.693$.

4.25 $x = 2.$

4.26 $\gamma = \frac{1}{T}\,\ln\frac{x(t)}{x(t+T)} = 100\,\ln 2.$

4.27

Grad	$\underline{40,36°}$	$81,19°$	$-322,08°$	$\underline{278,19°}$
Bogen	$0,7044$	$\underline{1.4171}$	$\underline{-5.6213}$	4.8553

4.28 $\cos(x_1 - x_2) = \cos x_1\,\cos x_2 + \sin x_1\,\sin x_2$
$x_1 = x_2 = x \Rightarrow \cos(0) = 1 = \cos^2 x + \sin^2 x$

4.30

	Amplitude	Phasenverschiebung	Periode
a)	2	$-\frac{\pi}{18}$	$\frac{2}{3}\pi$
b)	5	2.1	π
c)	10	-3	2
d)	2.4	$-\frac{\pi}{8}$	$\frac{1}{2}\pi$

4.32 $\pi/2,\ \pi/4,\ -\pi/3,\ 0.5018,\ \pi/3,\ \pi/6,\ \pi,\ 0.5489,\ \pi/4,\ -\pi/3,\ 2\pi/3,\ \pi/3$

4.33 $0.7071,\ 0.9792,\ 0.5225,\ -4.455,\ 0.8776$

4.34 $y = \arccos(x) \hookrightarrow x = \cos y$
$\sqrt{1-x^2} = \sqrt{1-\cos^2 y} = \sin y = \sin(\arccos(x))$

4.35 analog 6.8.

4.36 $x,\ x,\ \sqrt{1-x^2},\ \sqrt{1-x^2},\ x/\sqrt{1+x^2},\ \sqrt{1-x^2}/x$

4.37 a) $\mathbb{D}=[-1,1]$, $\mathbb{W}=[1,\pi-1)$,
b) $\mathbb{D}=[0,1]$, $\mathbb{W}=[0,\pi/2+1]$,
c) $\mathbb{D}=[0,2]$, $\mathbb{W}=[0,\pi]$

A.5 Lösungen zu komplexen Zahlen

A.5

5.1 a) $6\, e^{i\,\frac{\pi}{6}}$ b) $2\sqrt{2}\, e^{i\,\frac{5}{4}\,\pi}$ c) $2\, e^{i\,\frac{5}{3}\,\pi}$ d) $5\, e^{0\,i}$ e) $5\, e^{i\,\frac{3}{2}\,\pi}$ f) $e^{i\,\pi}$

5.2 a) $3\sqrt{2}\,\left(\cos\frac{\pi}{4} + i\,\sin\frac{\pi}{4}\right) = 3 + 3\,i$
b) $2\,\left(\cos\frac{2}{3}\,\pi + i\,\sin\frac{2}{3}\,\pi\right) = -1 + \sqrt{3}\,i$
c) $1\,\left(\cos\pi + i\,\sin\pi\right) = -1$
d) $4\,\left(\cos\frac{4}{3}\,\pi + i\,\sin\frac{4}{3}\,\pi\right) = -2 - 2\sqrt{3}\,i$

5.3 a) $3 - \sqrt{2}\,i$ b) $4\,(\cos 125° - i\,\sin 125°)$ c) $5\, e^{-i\,\frac{3}{2}\,\pi}$ d) $\sqrt{3}\, e^{-i\,0.734}$

5.4 a) $2\,\left(\cos\frac{2}{3}\,\pi + i\,\sin\frac{2}{3}\,\pi\right)$
b) $\sqrt{2}\,(\cos 135° + i\,\sin 135°)$
c) $2(\cos 45° + i\,\sin 45°)$
d) $5(\cos 233.13° + i\sin 233.13°)$

5.5 a) $1 - 4\,i$ b) $-9 - 46\,i$ c) $\frac{11}{5} - \frac{2}{5}\,i$ d) -1 e) $\frac{10}{37}$ f) $\frac{16}{5} - \frac{2}{5}\,i$

5.6 a) $-1 - 4\,i$ b) 170 c) $-1024\,i$ d) 12 e) $\frac{3}{5}$
f) $-\frac{1}{7}$ g) $-7 + 3\sqrt{3} + \sqrt{3}\,i$ h) $765 + 128\sqrt{3}$ i) $\frac{(6\sqrt{3}+4)}{7}$

5.7 a) $-512 + 512\sqrt{3}\,i$ b) $8\,(\cos 135° + i\,\sin 135°)$ c) $-46\,656$
d) $2^7\, e^{i\,1.66\,\pi} = 2\, e^{i\,5.21}$

5.8 a) $3\, e^{i\varphi}$ mit $\varphi = \frac{\pi}{4},\ \frac{3}{4}\,\pi,\ \frac{5}{4}\,\pi,\ \frac{7}{4}\,\pi$
$3\,(\cos\varphi + i\,\sin\varphi)$ mit $\varphi = 45°,\ 135°,\ 225°,\ 315°$
b) $\sqrt[6]{2}\, e^{i\varphi}$ mit $\varphi = \frac{\pi}{9},\ \frac{4}{9}\,\pi,\ \frac{7}{9}\,\pi,\ \frac{10}{9}\,\pi,\ \frac{13}{9}\,\pi,\ \frac{16}{9}\,\pi$
$\sqrt[6]{2}\,(\cos\varphi + i\,\sin\varphi)$ mit $\varphi = 20°,\ 80°,\ 140°,\ 200°,\ 260°,\ 320°$

5.9 a) $1,\ 1,\ 2,\ -1 \pm i$
b) $1, -2,\ \frac{1}{2}\,i, -\frac{1}{2}\,i$

5.11 evalc (Re((-2+7*I) / (15*I))); etc.
evalc (Im((-2+7*I) / (15*I))); etc.

5.12 abs (") → Betrag
argument (") → Winkel

5.13 evalc

5.14 evalc

5.15 solve (z ^3 = I, z) bzw. fsolve (..., z)

5.16 fsolve (" , z, complex)

5.17 a) $\hat{R} = 100\,\Omega + i\,(199\,999.95)\,\Omega \Rightarrow R = \left|\hat{R}\right| = 199\,999.98\,\Omega$
b) $\hat{R} = 86.21\,\Omega + i\,34.48\,\Omega \Rightarrow R = \left|\hat{R}\right| = 92.85\,\Omega$

5.18 a) $\hat{R} = R_1 + \frac{(\omega\,L)^2 \cdot R_2}{(\omega\,L)^2 + R_2^2} + i\,\frac{\omega\,L\,R_2^2}{(\omega\,L)^2 + R_2^2}$ b) $\hat{R}_g = 609.36\,\Omega + i\,497.11\,\Omega$

5.19 $u = 231.77\,V \cdot \sin(\omega t + 0.48)$

5.20 $y = 22.37\,cm \cdot \cos(\omega t + 5.74)$

A.6

A.6 Lösungen zu Grenzwert und Stetigkeit

6.1 a) $n > 10^3$ b) $n > 10^4$ c) $n \geq 10^{10}$

6.2 a) $\frac{1}{2}$ b) ∞ c) 1 d) 5 e) $\frac{1}{2}$ f) $\frac{4}{3}$ g) $\frac{4}{5}$ h) 3 i) $\sin\frac{\pi}{2} = 1$

6.3 $\left|a_n - \frac{1}{2}\right| < \varepsilon \Leftrightarrow n > \frac{1}{4\,\varepsilon}$

6.4 a) $\frac{1}{2}$

b) Mit $q := \left|\frac{a_{n+1}}{a_n}\right|$ folgt $|a_n| \leq q^{n-1}\,|a_1| \to 0$ für $n \to \infty$.

6.5 $L1 : |a_n + b_n - (a+b)| \leq |a_n - a| + |b_n - b| \overset{n\to\infty}{\longrightarrow} 0$

$L2 : |a_n \cdot b_n - a \cdot b| = |a_n\,(b_n - b) + (a_n - a)\,b|$

$\leq |a_n|\,|b_n - b| + |b|\,|a_n - a| \overset{n\to\infty}{\longrightarrow} 0$

6.6 a) limit (1 / n * sum(1 / i, i = 1..n), n = infinity);

b) limit (n / sqrt[n] (n!), n = infinity);

6.7 a) 7 b) $-\frac{2}{3}$ c) 0

6.8 a) 0 b) -7 c) 2 d) $\frac{7}{4}$ e) $\frac{1}{2}$ f) 1

6.9 $\lim\limits_{x\to 1} f(x) = 2$

6.10 $\lim\limits_{h\to 0} f(x-h) = 0 \neq \lim\limits_{h\to 0} f(x+h) = -2$

6.11 $\lim\limits_{h\to 0} f(x_0+h) = 2 = f(x_0) = \lim\limits_{h\to 0} f(x_0-h)$

6.12 Ja: $\tilde{f}(1) := \frac{1}{2}$

A.7

A.7 Lösungen zur Differenzialrechnung

7.1 a) $y' = 56\,x^6 - 30\,x^2 - 30\,x^{-4} + 56\,x^{-8}$

b) $y' = 9\,x^{-\frac{1}{4}} - 4\,x^{-\frac{3}{7}} + 11 + 12\,x^{-\frac{5}{2}}$

c) $y' = \frac{1}{60}\,l^{-\frac{59}{60}}$

d) $y' = \frac{-13\cdot 9}{2}\,a^{-\frac{15}{2}}$

e) $y' = 2\,b\,x\,(c+e\,x)^3 + \left(a + b\,x^2\right)\cdot 3\,(c+e\,x)^2 \cdot e$

f) $y' = \frac{7}{2}\,x^{\frac{5}{2}} + \frac{5}{2}\,x^{\frac{3}{2}}$

g) $y' = (\alpha+\beta)\,x^{\alpha+\beta-1}$

7.2 a) $\frac{10\cos(x)}{x^3} - \frac{30\sin(x)}{x^4}$

b) $\cos^2(x) - \sin^2(x)$

c) $nx^{n-1}e^x + x^n e^x$

d) $\frac{-7}{(x-10)^2}$

e) $\frac{-1}{1-\cos(\varphi)}$

f) $-4\frac{2t^2-t+1}{(t+1)^2(t-1)(t^2-1)}$

g) $-\frac{x^2-2}{(x^2+2)^2}$

h) $-\frac{xe^x(-2e^x+2+x)}{(e^x-1)^2}$

i) $-\frac{\ln(x)+1-x+x\ln(x)}{(x-1)^3}$

7.3 a) $y'(x) = -\sin(3\,x+2)\cdot 3$

b) $y'(x) = 3\,(3\,x-2)^2 \cdot 3$

c) $y'(x) = 15\,\cos(5\,x)$

d) $y'(x) = (8\,x-3)\,e^{4\,x^2-3\,x+2}$

e) $y'(x) = \frac{20\,x}{1+x^2}$

f) $x'(t) = A\,\omega\,\cos(\omega t + \varphi)$

g) $y'(x) = \frac{\cos(2\,x-3)}{\sin(2\,x-3)} \cdot 2$

h) $y'(x) = \frac{1}{2}\,\frac{1}{\sqrt{\ln(x^2-1)}} \cdot \frac{2\,x}{x^2-1}$

7.4 a) $y'(x) = x^x\,(\ln x + 1)$

b) $y'(x) = x^{\sin x} \cdot \frac{x\,\cos x\,\ln x + \sin x}{x}$

7.5 a) $y' = x^{(x^x)} \cdot x^x\,\left((\ln x + 1)\,\ln x + \frac{1}{x}\right)$

b) $y' = (x^x)^x \cdot x\,(2\,\ln x + 1)$

c) $y' = x^{(x^a)+a-1}\,(a\,\ln x + 1)$

d) $y' = x^{(a^x)}\,a^x\,\left(\ln a\,\ln x + \frac{1}{x}\right)$

e) $y' = a^{(x^x)} \cdot x^x\,(\ln x + 1)\,\ln a$

7.6 a) $\frac{t}{t^2-a^2}$ b) $\frac{2\,x}{x^4-1}$ c) $\frac{x+13}{x^2-4\,x-5}$ d) $\frac{a^{\ln(x-3)}\,\ln a}{x-3}$ e) $\frac{e^x\,(2-x^2)}{(1-x)\,\sqrt{1-x^2}}$ f) 1

7.7 $\sinh'(x) = \cosh(x)$, $\cosh'(x) = \sinh(x)$, $\tanh'(x) = \frac{1}{\cosh^2(x)}$

7.8 $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$, $\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}$

$\arctan'(x) = \frac{1}{x^2+1}$, $\operatorname{arccot}'(x) = -\frac{1}{x^2+1}$

7.9 $ar\sinh'(x) = \frac{1}{\sqrt{x^2+1}}$, $ar\cosh'(x) = \frac{1}{\sqrt{x^2-1}}$

7.10 $y(x) = x^n \hookrightarrow \ln y = n\,\ln x \hookrightarrow y' = y \cdot n\,\frac{1}{x} = n\,x^{n-1}$

7.11 a) $\frac{-2\,\sin(2\,x) - e^{x\,y} \cdot y - \frac{y^3}{x}}{e^{x\,y} \cdot x + 3\,y^2\,\ln x}$

b) $\frac{y \cdot \ln y \cdot e^{x\,y} - \frac{4 \cdot \ln a}{y^3}}{\frac{e^{-x\,y}}{y} - x \cdot \ln y \cdot e^{-x\,y} - \frac{12\,x+6}{y4}}$

c) $\frac{2\,y}{y - 2\sqrt{y}}$

d) $\frac{2\,x\,y}{\cos y - x^2}$

7.12 $y'(4) = -0.436$

7.13

	f	$\sqrt{1+x^4}$	$3\,\ln(1+3\,x^5)$	$2\,\cos x$
	f'	$2\,\frac{x^3}{\sqrt{1+x^4}}$	$\frac{45\,x^4}{1+3\,x^5}$	$-2\,\sin x$
i)	$df = f'(x)\,dx$	$2\,\frac{x^3}{\sqrt{1+x^4}}\,dx$	$\frac{45\,x^4}{1+3\,x^5}\,dx$	$-2\,\sin x\,dx$
ii)	$df(x_0) = f'(x_0)\,dx$	$\sqrt{2}\,dx$	$4.993\,dx$	$\sqrt{2}\,dx$
iii)	$y_t = f(x_0) + df(x_0)$	$\sqrt{2}\,x$	$4.993\,x + 4.8$	$1.414\,x - 0.3034$
iv)	*Linearisierung*	$\sqrt{2}\,x$	$4.993\,x + 4.8$	$1.414\,x + 0.3034$
v)	$f(x_0 + \varepsilon) \approx$	1.4283	19.8289_3	1.428
	exakt	1.4142	19.8289_8	1.400
	Punkt $(x_0,\,y_0)$	$(1,\,\sqrt{2})$	$(3,\,19.779)$	$(\frac{\pi}{4},\,\sqrt{2})$

7.14 a) $x(t) = A\,e^{-\gamma\,t}\,\cos(\omega t)$

$\dot{x}(t) = -\gamma\,A\,e^{-\gamma\,t}\,\cos(\omega t) - \omega\,A\,e^{-\gamma\,t}\,\sin(\omega t)$

$\ddot{x}(t) = A\,\gamma^2\,e^{-\gamma\,t}\,\cos(\omega t) + A\,\gamma\,\omega\,e^{-\gamma\,t}\,\sin(\omega t) -$

$A\,\omega^2\,e^{-\gamma\,t}\,\cos(\omega t) + A\,\gamma\,\omega\,e^{-\gamma\,t}\,\sin(\omega t)$

b) $\dot{x}(t) = 0 \Rightarrow -\gamma\,\cos(\omega t) - \omega\,\sin(\omega t) = 0 \Rightarrow \tan(\omega t) = -\frac{\gamma}{\omega}$

7.15 $V'(a) = 0$ mit $V''(a) = \frac{2\,D}{a^2} > 0$

7.16 relativer Fehler $\frac{d\alpha}{\alpha} = 3.9 \cdot 10^{-3} \approx 4\,{}^0/_{00}$

7.17 a) Minimum $\left(-\frac{1}{2};-5\right)$ Maximum (1.5; 27)

b) Maximum (0; 16) Minimum (±2; 0)

c) Maximum (0; 2)

d) Maximum (1; 0.368)

e) Maxima $x_k = \frac{\pi}{4} + k \cdot \pi$ $y_k = 0.5$ $k \in \mathbb{Z}$

Minima $x_k = \frac{3}{4}\pi + k \cdot \pi \quad y_k = -0.5 \quad k \in \mathbb{Z}$

f) Minimum $(0,5; -0.08)$

7.18 a) $y = \frac{x^2+1}{x-3}$: $\mathbb{D} = \mathbb{R} \setminus \{3\}$, $\mathbb{W} = (-\infty, -0.325] \cup 12.325, \infty)$,
Pol: $x = 3$, Vertikale Asymptote $x = 3$, Asymptote im Unendlichen $y = x + 3$,
Extremwerte: Max $(-0.162, -0.325)$ Min $(6.162, 12.325)$.

b) $y = \frac{(x-1)^2}{x+1}$: $\mathbb{D} = \mathbb{R} \setminus \{1\}$, $\mathbb{W} = (-\infty, -8] \cup [0, \infty)$,
Pol: $x = -1$, Vertikale Asymptote $x = -1$; Asymptote im Unendlichen $y = x - 3$,
Extremwerte: Max $(-3, -8)$ Min $(1, 0)$.

c) $y = \frac{\ln x}{x}$: $\mathbb{D} = (0, \infty)$, $\mathbb{W} = (-\infty, 0.368)$,
Nullstellen: $x = 1$, Pol: $x = 0$, Asymptote für $x \to \infty$: $y = 0$,
Extremwert: Max $(2.71, 0.368)$, Wendepunkt: $(4.48, 0.335)$.

d) $y = \sin^2 x$: $\mathbb{D} = \mathbb{R}$, $\mathbb{W} = [0, 1]$,
Periodizität π, Nullstellen: $x_k = k\,\pi$,
Extrema: Max $\left(x_k = \frac{\pi}{2} + k\,\pi;\quad y_k = 1\right)$ Min $(x_n = k\,\pi;\quad y_k = 0)$,
Wendepunkte $x_k = \frac{\pi}{4} + k \cdot \frac{\pi}{2} \quad y_k = \frac{1}{2}$.

7.19 a) $2\,a$ b) 2 c) 2 d) $\frac{1}{12}$ e) 0 f) $\frac{1}{-2}$ g) 1 h) e^a

A.8 Lösungen zur Integralrechnung

8.1 rightbox (sqrt(x), x = 0..2, 10); rightsum (sqrt(x), x = 0..2, 10);

8.2 a) $\frac{x^6}{6} + C$
b) $-\frac{1}{x} + C$
c) $\frac{3}{4}\,z^{\frac{4}{3}} + C$
d) $3\,x^{\frac{1}{3}} + C$
e) $\frac{2}{3}\,x^3 - \frac{5}{2}\,x^2 + 3\,x + C$
f) $\frac{2}{3}\,x^{\frac{3}{2}} - \frac{2}{5}\,x^{\frac{5}{2}} + C$

8.3 a) 1 b) $2\,\pi^2 + 2$ c) $\ln a$

8.4 a) $x\,\sin x + \cos x + C$
b) $\frac{1}{2}\,\sin^2 x + C$
c) $-x^2\,\cos x + 2\,x\,\sin x + 2\,\cos x + C$
d) $\frac{1}{3}\,x^3\,\ln x - \frac{1}{9}\,x^3 + C$
e) $x\,e^x - e^x + C$
f) $x^2\,e^x - 2\,x\,e^x + 2\,e^x + C$

8.5 a) $\ln|x + 2| + C$ b) $\frac{1}{2}\,\ln\left|x^2 - 1\right| + C$ c) $-\frac{1}{6}\,\ln\left|1 - 2\,x^3\right| + C$
d) $\frac{1}{9}\,\frac{1}{3}\,(3s + 4)^9 + C$ e) $-\frac{1}{\omega}\,\cos(\omega t + \varphi) + C$ f) $\frac{1}{3}\,\sin(3t) + C$
g) $-e^{-x} + C$ h) $-\ln|\cos t| + C$ i) $\ln|x\,e^x| + C$
j) $\frac{1}{2}\,\sin^2 x + C$ k) $\frac{2}{3}\,\frac{1}{3}\,(4 + 3\,x)^{\frac{3}{2}} + C$

8.6 Nachrechnen durch Differenzieren der rechten Seite

8.7 a) $-\frac{2}{3}\,x\,\sqrt{x} + x + 2\,\sqrt{x} - 2\,\ln(1 + \sqrt{x}) + C$ b) $-\frac{1}{3}\,\sqrt{1 - x^2}^3 + C$

8.8 a) $\frac{2}{3}\sqrt{1+x^3}+C \quad \left(u=1+x^3\right)$
b) $\frac{2}{15}\sqrt{5\,x+12}^3+C \quad (u=5\,x+12)$
c) $-\frac{3}{4}\sqrt[3]{(1-t)^4}+C \quad (u=1-t)$
d) $0 \quad (u=\cos x)$
e) $\frac{1}{2}\arctan^2(z)+C \quad (u=\arctan z)$
f) $\ln\left|x^2+6\,x-12\right|+C \quad \left(u=x^2+6\,x-12\right)$
g) $\ln|\ln(x)|+C \quad (u=\ln x)$
h) $-\frac{1}{2}\cos\left(x^2\right)+C \quad \left(u=x^2\right)$
i) $\frac{1}{2}\ln\left|2\,x^3-4\,x+2\right|+C \quad \left(u=2\,x^3-4\,x+2\right)$
j) $0 \quad \left(u=1+t^2\right)$
k) $0.47 \quad \left(u=3\,t-\frac{\pi}{4}\right)$
l) $2.055 \quad (u=5-x)$
m) $\frac{1}{3}e^{x^3-2}+C \quad \left(u=x^3-2\right)$
n) $\frac{1}{2}\tan^2(z+5)+C \quad (u=\tan(z+5))$
o) $-\frac{\sqrt{4-x^2}}{x}-\arcsin\left(\frac{x}{2}\right)+C \quad (x=2\sin u)$

8.9 a) $\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$
b) $x\cdot\sin x+\cos x+C$
c) $t\ln t-t+C$
d) $-\frac{1}{3}x\cos(3\,x)+\frac{1}{9}\sin(3\,x)+C$
e) $x\arctan x-\frac{1}{2}\ln\left(1+x^2\right)+C$
f) $\frac{1}{2}\left(t-\frac{1}{\omega}\sin(\omega t)\cos(\omega t)\right)$
g) $\frac{1}{2}e^x(\sin x+\cos x)+C$
h) $-x^2e^{-x}-2\,x\,e^{-x}-2\,e^{-x}+C$

8.10 a) $\frac{1}{2\,a}(\ln|x-a|-\ln|x+a|)+C$
b) $2\ln|x+1|+\frac{2}{3}\ln|x-1|-\frac{32}{3}\ln|x+2|+C+4\,x$
c) $\frac{1}{3}\ln\left|\frac{z-1}{z+2}\right|-2\,\frac{1}{z+2}+C$
d) $\frac{17}{8}\ln|x-9|+\frac{15}{8}\ln|x+7|+C$
e) $\frac{1}{9}\ln\left|\frac{x}{x-3}\right|-\frac{7}{3\,(x-3)}+C$

8.11 a) $\frac{2}{3}(\ln x)^{\frac{3}{2}}+C$
b) $\ln|\sin x|+C$
c) $x\sinh(x)-\cosh(x)+C$
d) $-e^{\cos x}+C$
e) $x+\frac{1}{4}\ln|x-1|-\frac{5}{4}\ln|x+1|-\frac{1}{2}\frac{1}{x+1}+C$
f) $x-5\ln|x+1|$
g) $\frac{1}{4}(\ln x)^4+C$
h) $2\ln\left|2\,x^3-1\right|+C$ i) $\frac{1}{2}(x^2+1)\arctan x-\frac{1}{2}x+C$

8.12 $\ln(x+\sqrt{1+x^2})\,x-\sqrt{1+x^2}$

8.13 $\frac{8}{3}-\frac{4}{3}\sqrt{1+\sqrt{x}}(2-\sqrt{x})\,+C$

8.14 convert (" , x, parfrac)

8.15 a) 0 b) 0 c) 0 für $n\neq m$; π für $n=m$
d) 0 für $n\neq m$; π für $n=m$ e) 0

8.16 a) $\frac{1}{2}$ b) $\frac{\pi}{2}$ c) $\frac{1}{2}$ d) $\frac{1}{-a+s}$ e) $\frac{s}{s^2+a^2}$ f) $\frac{n!}{s^{n+1}}$

A.9

A.9 Lösungen zu Funktionenreihen

9.1 a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \Rightarrow$ Satz: Divergenz
b) Quotientenkriterium $\Rightarrow$ Konvergenz
c) $\frac{|\sin n|}{n^2} \leq \frac{1}{n^2} \Rightarrow$ Majorantenkriterium: Konvergenz
d) $\frac{n}{2\,n+1} \underset{n\to\infty}{\longrightarrow} \frac{1}{2} \Rightarrow$ Koeffizienten keine Nullfolge $\Rightarrow$ Divergenz
e) Quotientenkriterium $\Rightarrow$ Konvergenz
f) Quotientenkriterium $\Rightarrow$ Konvergenz
g) Quotientenkriterium $\Rightarrow$ Konvergenz
h) Leibniz-Kriterium $\Rightarrow$ Konvergenz
i) Leibniz-Kriterium $\Rightarrow$ Konvergent
j) Quotientenkriterium $\Rightarrow$ Konvergenz
k) Quotientenkriterium $\Rightarrow$ Divergenz
l) Majorantenkriterium $\leq \frac{1}{n^2}$

9.2 a) 6 b) e c) 6

9.4 5, 1, $\frac{1}{e}$, 4

9.5 a) $K = (-2,\, 2)$ b) $K = [-1,\, 1]$ c) $K = (-1,\, 1)$ d) $K = (-1,\, 1]$
e) $K = (-2,\, 2)$ f) $K = (-1,\, 1)$ g) $K = \mathbb{R}$ h) $K = \left[-\frac{1}{2},\, \frac{1}{2}\right)$

9.6 a) $K = (-e+4,\, e+4)$ b) $(-1,\, 3)$ c) $K = \mathbb{R}$ d) $K = \mathbb{R}$

9.7 siehe §9.3 Tabelle 9.1

9.8 $f(x) = -1 + (x-1)^2 - 2\,(x-1)^3 + 3\,(x-1)^4 - \ldots \pm (n-1)\,(x-1)^n \pm \ldots$
$= -1 + \sum_{n=2}^{\infty} (n-1)\,(x-1)^n\,(-1)^{n+1}; \quad K = (0,\, 2]$

9.9 siehe §9.3 Tabelle 9.1

9.10 siehe §9.3 Tabelle 9.1

9.11 siehe §9.3 Tabelle 9.1

9.12 $f(x) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\,n)!} \left(x - \frac{\pi}{3}\right)^{2\,n} + \frac{1}{2}\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2\,n+1)!} \left(x - \frac{\pi}{3}\right)^{2n+1}$
$K = \mathbb{R}$

9.14 $f(x) = x - x^2 + \frac{1}{2}\,x^3 + R_3(x)$ mit $|R_3(x)| \leq \frac{1}{6}\,|x|^4$

9.15 $R_n(x) = \frac{1}{n!}\,\frac{1}{2}\,\frac{1\cdot 3\cdot\ldots\cdot(2\,n-3)}{2^n}\,(1-\xi)^{-\frac{2\,n-1}{2}}\,x^n \leq 10^{-4}$ für $\xi \in (0,\, 0.05)$
$\Rightarrow n = 5$

9.18 $F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1}\,x^{2\,n+1}$

9.19 taylor (m / k * ln(cosh(sqrt(k * g / m) * t)), k = 0, 3);
$\frac{1}{2}\,g\,t^2 - \frac{1}{12}\,\frac{g^2\,t^4}{m}\,k + \frac{1}{45}\,\frac{g^3\,t^6}{m^2}\,k^2 + O\left(k^3\right)$

9.20 $\mathrm{sinc}(x) = x - \frac{x^3}{3!\,3} + \frac{x^5}{5!\,5} - \frac{x^7}{7!\,7} \pm \ldots$
$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}} \left(\frac{x}{1} - \frac{x^3}{1!\,3} + \frac{x^5}{2!\,5} - \frac{x^7}{3!\,7} \pm \ldots\right)$

9.21 a) $z^3 = x^3 - 3\,x\,y^2 + i\,\left(3\,x^2\,y - y^3\right)$
b) $\frac{1}{1-z} = \frac{1-x}{(1-x)^2+y^2} + i\,\frac{y}{(1-x)^2+y^2}$
c) $e^{3\,z} = e^{3\,x}\,\cos 3\,y + i\,e^{3\,x}\,\sin 3\,y$

9.22 $\left|e^{i\,z}\right| = 1\,e^{-3\sqrt{3}}$

9.23 a) $f_i'(x) = 3\,(1+i)^3\,x^2; \quad f_{ii}'(x) = 3\,(1+i)\,e^{3\,(1+i)\,x}$
b) $\int f_i(x)\,dx = (1+i)^3\,\frac{1}{4}\,x^4 + c; \quad \int f_{ii}(x)\,dx = \frac{1}{3\,(1+i)}\,e^{3\,(1+i)\,x} + c$

9.24 $\hat{R}_L = iwL$

A.10 Lösungen zur Differenzialrechnung bei Funktionen mit mehreren Variablen

A.10

10.1 plot3d(z, x=a..b, y=c..d);

10.2 plot3d(z, x=a..b, y=c..d, style=contour, contours=20);

10.3 a) $f_x = 3\,x^2 + y; \quad f_y = x + 2\,y^{-3}$

b) $f_a = 3\,x; \quad f_t = 2\,y\,t^{-1}$

c) $f_u = \frac{(u+v)-(u+w)}{(u+v)^2}; \quad f_v = -\frac{u+w}{(u+v)^2}$

d) $f_x = \frac{2\,x}{\sqrt{1+\left(x^2+z^2\right)^2}}; \quad f_y = 0; \quad f_z = \frac{2\,z}{\sqrt{1+\left(x^2+z^2\right)^2}}$

e) $f_{x_1} = f_{x_3} = 0; \quad f_{x_2} = 1$

f) $f_a = -\left(a\,x + b\,x^2\right)^{-2}\,x + y\,b\,e^{a\,b}; \quad f_b = -\left(a\,x + b\,x^2\right)^{-2}\,x^2 + y\,a\,e^{a\,b}$

10.4 a) $f_x = 6\,x + 4\,y; \quad f_y = 4\,x - 4\,y; \; f_{xx} = 6; \quad f_{xy} = f_{yx} = 4; \quad f_{yy} = -4$

b) $f_x = -6\,y\,\sin(3\,x\,y); \quad f_y = -6\,x\,\sin(3\,x\,y)$

$f_{xx} = -18\,y^2\,\cos(3\,x\,y); \quad f_{xy} = f_{yx} = -6\,\sin(3\,x\,y) - 18\,x\,y\,\cos(3\,x\,y);$

$f_{yy} = -18\,x^2\,\cos(3\,x\,y)$

c) $f_x = 12\,(3\,x - 5\,y)^3; \quad f_y = -20\,(3\,x - 5\,y)^3$

$f_{xx} = 108\,(3\,x - 5\,y)^2; f_{xy} = f_{yx} = -180\,(3\,x - 5\,y)^2; f_{yy} = 300\,(3\,x - 5\,y)^2$

d) $f(x,\,y) = x - y; \; f_x = 1; \; f_y = -1; \; f_{xx} = 0; \; f_{xy} = f_{yx} = 0; \; f_{yy} = 0$

e) $f_x = 3\,e^{x\,y} + 3\,x\,y\,e^{x\,y}; \; f_y = 3\,x^2\,e^{x\,y}$

$f_{xx} = 6\,y\,e^{x\,y} + 3\,x\,y^2\,e^{x\,y}; \; f_{xy} = f_{yx} = 6\,x\,e^{x\,y} + 3\,x^2\,y\,e^{x\,y}; \; f_{yy} = 3\,x^3\,e^{x\,y}$

f) $f_x = \frac{x-y}{\left(x^2-2\,x\,y\right)^{1/2}}; \quad f_y = \frac{-x}{\left(x^2-2\,x\,y\right)^{1/2}}$

$f_{xx} = \frac{-y^2}{\left(x^2-2\,x\,y\right)^{3/2}}; \quad f_{xy} = f_{yx} = \frac{x\,y}{\left(x^2-2\,x\,y\right)^{3/2}}; \quad f_{yy} = \frac{-x^2}{\left(x^2-2\,x\,y\right)^{3/2}}$

10.5 $\frac{\partial}{\partial x}\,f(x,\,y) = 2\,x\,\cos\left(x^2 + 2\,y\right); \quad \frac{\partial}{\partial y}\,f(x,\,y) = 2\,\cos\left(x^2 + 2\,y\right)$

$\frac{\partial}{\partial y}\,\frac{\partial}{\partial x}\,f(x,\,y) = 2\,x\cdot\left[-\sin\left(x^2+2\,y\right)\right]\cdot\left\{\frac{\partial}{\partial y}\,\left(x^2+2\,y\right)\right\} = -4\,x\,\sin\left(x^2+2\,y\right)$

$\frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\,f(x,\,y) = -2\,\sin\left(x^2+2\,y\right)\cdot\left\{\frac{\partial}{\partial x}\,\left(x^2+2\,y\right)\right\} = -4\,x\,\sin\left(x^2+2\,y\right)$

$\Rightarrow \frac{\partial}{\partial y}\,\frac{\partial}{\partial x}\,f(x,\,y) = \frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\,f(x,\,y)$

10.7 $f_{xx} = 108\,(3\,x - 5\,y)^2; f_{xy} = f_{yx} = -180\,(3\,x - 5\,y)^2; f_{yy} = 300\,(3\,x - 5\,y)^2$

10.8 $f_{xx} = \left(x^2+y^2+z^2\right)^{-5/2} \quad \left\{3\,a\,x^2 - a\,\left(x^2+y^2+z^2\right)\right\}$

$f_{yy} = \left(x^2+y^2+z^2\right)^{-5/2} \quad \left\{3\,a\,y^2 - a\,\left(x^2+y^2+z^2\right)\right\}$

$f_{zz} = \left(x^2+y^2+z^2\right)^{-5/2} \quad \left\{3\,a\,z^2 - a\,\left(x^2+y^2+z^2\right)\right\}$

$\Rightarrow f_{xx} + f_{yy} + f_{zz} = 0$

10.10 $f_{xx} = \frac{-x^2+y^2}{\left(x^2+y^2\right)^2} \quad ; \quad f_{yy} = \frac{x^2-y^2}{\left(x^2+y^2\right)^2} \Rightarrow f_{xx} + f_{yy} = 0$

10.11 $z_t = -9 + 18\,x + 6\,y$

10.12 $df = 0.5\,dx + 0.826\,dy$

10.13 $grad\,f = \begin{pmatrix} 2\,(3\,x + x\,y)\,(3+y) \\ 2\,x\,(3\,x + x\,y) \end{pmatrix}; \quad f_{\vec{a}} = \frac{2}{\sqrt{5}}\,x\,(3+y)^2 + \frac{4}{\sqrt{5}}\,x^2\,(3+y);$

→ gradplot

10.14 $grad\,f = \begin{pmatrix} \frac{2\,x}{\left(1+x^2\right)\,y} \\ \cos(z) - \frac{\ln\left(1+x^2\right)}{y^2} \\ -y\,\sin(z) \end{pmatrix}; \quad f_{\vec{a}} = \frac{2}{\sqrt{5}}\,\frac{x}{\left(1+x^2\right)\,y} - \frac{2}{\sqrt{5}}\,y\,\sin(z);$

→ gradplot3d

10.15 a) $dz = \left(12\,x^2\,y - 3\,e^y\right)\,dx + \left(4\,x^3 - 3\,x\,e^y\right)\,dy$

b) $dz = \frac{x^2-2\,x\,y-y^2}{(x-y)^2}\,dx + \frac{x^2+2\,x\,y-y^2}{(x-y)^2}\,dy$

c) $df = \frac{x}{x^2+y^2+z^2}\,dx + \frac{y}{x^2+y^2+z^2}\,dy + \frac{z}{x^2+y^2+z^2}\,dz$

10.16 a) $\frac{\partial}{\partial x_1}\,h\,(x_1,\,x_2) = c_1\,a_1;\quad \frac{\partial}{\partial x_2}\,h\,(x_1,\,x_2) = c_2\,a_2^2$

b) $\frac{\partial}{\partial x_1}\,h\,(x_1,\,x_2) = (2\,\sin(x_1) + \cos(x_2))\,\cos(x_1)\,;$

$\frac{\partial}{\partial x_2}\,h\,(x_1,\,x_2) = -\sin(x_1)\,\sin(x_2)$

10.17 a) $f\,(x,\,y) = \frac{1}{4}\,\left(y^2 - x^2\right) + (x-y) + R_2\,(x,\,y)$

b) $f\,(x,\,y) = e + 2\,e\,(x-1) + 3\,e\,(x-1)^2 + e\,y^2 + R_2\,(x,\,y)$

10.18 a) $df = 2\,x\,\cos\left(x^2 + 2\,y\right)\,dx + 2\,\cos\left(x^2 + 2\,y\right)\,dy$

b) $df = (6\,x + 4\,y)\,dx + (4\,x - 4\,y)\,dy$

c) $df = -y\,\sin(x - 2\,y)\,dx + (\cos(x - 2\,y) + 2\,y\,\sin(x - 2\,y))\,dy$

d) $df = \left(2\,x\,z + 4\,x^3\right)\,dx - z^3\,dy + \left(x^2 - 3\,y\,z^2\right)\,dz$

10.19 $dV = \frac{\partial V}{\partial d}\,dd + \frac{\partial V}{\partial h}\,dh; |dV| \le 0.6786$ (absoluter Fehler);

$\left|\frac{dV}{V}\right| \le 0.6\,\%$ (relativer Fehler)

10.20 $\frac{dR}{R} = 3.8\,\%$

10.21 $E = (212\,206 \pm 6030)\,\frac{N}{mm^2}$,

da $\quad dE = \frac{k}{\pi\,r^2\,z}\,dl - \frac{2\,l\,k}{\pi\,r^3\,z}\,dr + \frac{l}{\pi\,r^2\,z}\,dk - \frac{l\,k}{\pi\,r^2\,z^2}\,dz$

10.22 1.1 ist der relative Fehler.

10.23 $2 + \frac{1}{2}\,\ln 2 + \frac{1}{2}\,(x-1) + \left(1 - \frac{1}{4}\,\ln 2\right)\,(y-2)$

10.24 a) Stationäre Punkte: $(0,\,n\,\pi)\quad n \in \mathbb{Z}$, für n ungerade liegen lokale Minima vor.

b) Stationärer Punkt: $(0,\,0)$, Sattelpunkt

c) Stationäre Punkte: $(0,\,0)$ ist Sattelpunkt; $(1,\,-1)$ ist lokales Minimum.

10.25 $d = |P - z| = \left|\begin{pmatrix}1\\-2\\0\end{pmatrix} - \begin{pmatrix}x\\y\\\sqrt{1+(x-2\,y)^2}\end{pmatrix}\right|$

$= \sqrt{6 - 2\,x + 2\,x^2 - 4\,x\,y + 4\,y + 5\,y^2}$

$d \overset{!}{=}$ minimal$\hookrightarrow d_x = 0$ und $d_y = 0 \Rightarrow (x,\,y) = \left(\frac{1}{6},\,-\frac{1}{3}\right)$

10.27 $(1,\,2)$: relatives Minimum;

$(-1,\,-2)$: relatives Maximum;

$(-1,\,2)$ und $(1,-2)$: Sattelpunkte

10.28 a) $(0,\,0)$: Sattelpunkt ; $(1,\,1)$: relatives Maximum

b) $\left(-\frac{1}{2},\,\frac{1}{2}\right)$: relatives Minimum

c) $\left(\frac{1}{2},\,0\right)$: Sattelpunkt

d) $(0,\,0)$: Sattelpunkt ; $\left(0,\,\pm\sqrt{\ln 2}\right)$: lokales Minimum

10.29 Stationäre Punkte:

$P_1\,(0,\,0,\,0)\,;\ P_2\,(0,\,0,\,1)\,;P_3\,(0,\,0,\,-1)\,;P_4\,(0,\,\lambda,\,0)\,;$

$P_5\,(1,\,0,\,0)\,;P_6\,(-1,\,0,\,0)$.

Lokale Minima: P_2 , P_3 , P_5 , P_6

10.30 a) $-1.3150\,x + 2.0607$ b) $0.9619\,x + 0.4769$

A.11 Lösungen zur Integralrechnung bei Funktionen mit mehreren Variablen

A.11

11.1 a) $\frac{1}{3}\ln(l)$ b) -25 c) -2 d) $\frac{3}{2}\pi$

11.2 $\frac{1}{6}$

11.3 $I_1 = I_2 = \frac{32}{3}$

11.4 2π

11.5 a) $\frac{125}{6}$ b) $x_s = 2.5 \quad y_s = -2.5$

11.6 $I_x = I_y = \frac{\pi}{16} R^4 \qquad I_p = \frac{\pi}{8} R^4$

11.7 $x_s = \frac{2}{3} a \quad y_s = \frac{1}{3} b$

11.8 $x_s = 0 \quad y_s = \frac{4}{3\pi} R$

11.9 a) $\frac{1}{2}$ b) $\frac{4}{27} l^9$ c) $\frac{4}{3} R^3$ d) $\frac{2}{3} R^4$

11.10 $z_s = \frac{2}{3} R^2 \qquad I_x = I_y = \frac{1}{6} M R^2 \left(3 R^2 + 1\right) \qquad I_z = \frac{1}{3} M R^2$

11.11 $I_x = I_y = \frac{2}{5} M R^2 \quad I_z = \frac{2}{5} M R^2$

11.12 $I = \frac{1}{15}$

11.13 $x_s = y_s = \frac{4}{3\pi}, \quad z_s = \frac{1}{2}$

A.12 Lösungen zu Differenzialgleichungen

A.12

Differenzialgleichungen 1. Ordnung

12.1 a) $c\,e^{-4x}$ b) $c\,e^{-2x}$ c) $c\,e^{-\frac{8}{3}x}$ d) $c\,e^{\frac{b}{a}x}$ e) $c\,e^{6x}$ f) $c\,e^{-\frac{R}{L}t}$

12.2 a) $v(t) = 10\,\frac{m}{s} \cdot e^{-2t}, \quad t > 2.3\,\text{sec}$
b) $y(x) = e^{-\lambda x}, \quad \lambda = \ln 2$
c) $N(t) = N_0\, e^{-\frac{t}{\tau}}, \quad \tau = 8321.5$ Jahre!

12.3 a) $y_0\, e^{-\frac{1}{2}x^2} + 4$
b) $\frac{1}{4}\frac{1}{1+x}\left(4 y_0 + (2x+1)\right) e^{2x}$
c) $\frac{1}{x}\left(y_0 - x\cos x + \sin x\right)$
d) $\frac{1}{\cos x}\left(y_0 + x\right)$
e) $y_0\, e^{2\sin x} - \frac{1}{2}$
f) $x\left(y_0 + x - \frac{4}{x}\right)$

12.4 a) $y(t) = const \cdot e^{-\frac{1}{RC}t} + \frac{U_0 RC}{1+(\omega RC)^2}\sin(\omega t) + \frac{U_0 \omega R^2 C^2}{1+(\omega RC)^2}\cos(\omega t)$
b) $y(t) = const \cdot e^{-\frac{R}{L}t} + U_0\,\frac{L}{R-2L}\,e^{-2t}$

12.5 a) $\frac{x}{1+Cx}$
b) $c\sqrt{1+x^2}$
c) $\frac{e^{2x+2c}-1}{e^{2x+2c}+1}$
d) $1 - \frac{1}{x+C}$
e) $\arccos\left(\frac{1}{2}x^2 + C\right)$
f) $-\ln\left(-\sin x + C\right)$

12.6 a) $y(x) = 2\pi\, e^{-\sin x + 1}$
b) $y(x) = \frac{x}{x+1}$
c) $y(x) = \sqrt[3]{3 + 3x - x^3}$

d) $y(x) = \frac{x}{-1-\ln x}$
e) $y(x) = \sqrt{2 + 2\, e^{2\,x}}$

12.7 a) $y(x) = 4\,x\,\ln x + C\,x$ b) $y(x) = \frac{1}{2}\,x - \frac{x}{\ln C\,x}$

12.8 a) $y(x) = \sqrt[3]{\frac{1}{2}\,x^6 + C}$ b) $y(x) = C\,\frac{x-1}{x+1}$ c) $y(x) = C\,\sqrt{x^2+1}$
d) $y(x) = \sqrt{C \cdot e^x - 1}$ e) $y(x) = a\,\tan(x)$ f) $y(x) = \left(C - \sqrt{x^3}\right)^{-\frac{1}{3}}$

12.9 a) $y(x) = \frac{1}{\cos(x)}$ b) $y(x) = \cosh(x)$ c) $y(x) = \frac{\pi}{2}\,\sin(x)$
d) $y(x) = 2\,\left(x^2+1\right)$

Anwendungen DG 1. Ordnung

12.11 a) $x(t) = \frac{a\,b\left(e^{(a-b)\,k\,t}-1\right)}{b\,e^{(a-b)\,k\,t}-a}$
b) $\lim\limits_{t\to\infty} x(t) = \frac{a\,b}{a} = b$, d.h. wenn alle Moleküle vom Typ B reagiert haben.

12.12 a) $v(t) = \frac{m\,g}{k}\,\left(1 - e^{-\frac{k}{m}\,t}\right) + v_0\,e^{-\frac{k}{m}\,t}$
$s(t) = \frac{m\,g}{k}\,t + \left(\frac{m}{k}\right)^2\,g\,\left(e^{-\frac{k}{m}\,t} - 1\right) - v_0\,\frac{m}{k}\,\left(e^{-\frac{k}{m}\,t} - 1\right)$
b) $v_{\max} = \lim\limits_{t\to\infty} v(t) = \frac{m\,g}{k}$

12.13 $v(t) = \sqrt{\frac{m\,g}{k}}\,\tanh\sqrt{\frac{m\,g}{k}}\,t$
Für $t \to \infty$ wird die Endgeschwindigkeit erreicht $v_E = \lim\limits_{t\to\infty} v(t) = \sqrt{\frac{m\,g}{k}}$
$s(t) = \frac{v_E^2}{g}\,\ln\cosh\frac{g}{v_E}\,t$

12.14 $T(t) = e^{-a\,t}\,(T_0 - T_L) + T_L \Rightarrow \lim\limits_{t\to\infty} T(t) = T_L$

12.15 $v_z(r) = -\frac{1}{4\,\eta}\,\frac{\Delta p}{\Delta z}\,\left(R^2 - r^2\right)$ (Gesetz von Hagen-Poiseuille)

12.16 a) $v(t) = \frac{1}{R}\cdot m\cdot g\cdot\sin\varphi\cdot\left(1 - e^{-\frac{R}{m}\,t}\right)$
b) $v(t) = \sqrt{\frac{m\,g\,\sin\varphi}{D}}\,\tanh\left(\sqrt{\frac{D\,g\,\sin\varphi}{m}}\,t\right) = \left(\frac{5}{2}\,\tanh(2\,t)\right)$
c) $v(t) = \frac{1}{8}\,\left(25\,\tanh\left(\ln 2 + \frac{5}{2}\,t\right) - 15\right)$

Lineare Differenzialgleichungssysteme

12.17 a) $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^t, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{4\,t}, \quad \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} e^{6\,t}$
b) $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} e^t, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} e^{-t}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{2\,t}$

12.18 $\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{\sqrt{0}\,t}, \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{\pm\sqrt{5}\,t}$

12.19 a) $\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\omega t}, \quad \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-i\omega t}$ komplexes Fundamentalsystem
$\begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}, \quad \begin{pmatrix} \sin(\omega t) \\ -\cos(\omega t) \end{pmatrix}$ reelles Fundamentalsystem
b) $\psi(t) = E_0 \begin{pmatrix} -\frac{1}{\omega}\,t \\ \frac{1}{\omega^2} \end{pmatrix}$ spezielle Lösung

12.20 a) $-2\ EW \hookrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}\ EV \qquad -4\ EW \hookrightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}\ EV$

b) $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}, \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$ FS

c) $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\sqrt{2}\,i\,t}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\sqrt{2}\,i\,t}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2\,i\,t}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2\,i\,t}$ kompl. FS

d) $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\sqrt{2}\,t, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin\sqrt{2}\,t, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos 2\,t, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin 2t$ reelles FS

e) $\vec{y}'(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -3 & 0 \end{pmatrix} \vec{y}(t)$

12.21 a) $\det(A - \lambda I) = \lambda^2 + 2\lambda + 5 = (-1 + 2i - \lambda)(-1 - 2i - \lambda)$

$\vec{\varphi}_1(x) = \begin{pmatrix} -\frac{4}{5} - \frac{2}{5}i \\ 1 \end{pmatrix} e^{(-1+2i)x}, \quad \vec{\varphi}_2(x) = \begin{pmatrix} -\frac{4}{5} + \frac{2}{5}i \\ 1 \end{pmatrix} e^{(-1-2i)x}$

b) $\det(A - \lambda I) = (3 - \lambda)(2 - \lambda)(6 - \lambda)$

$\vec{\varphi}_1(x) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2x}, \quad \vec{\varphi}_2(x) = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} e^{3x}, \quad \vec{\varphi}_3(x) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{6x}$

c) $\det(A - \lambda I) = (2 - \lambda)^2 (5 - \lambda)$

$\vec{\varphi}_1(x) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2x}, \quad \vec{\varphi}_2(x) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{2x}, \quad \vec{\varphi}_3(x) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5x}$

12.22 $\vec{y}'(x) = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 4 \end{pmatrix} \vec{y}(x), \quad \det(A - \lambda I) = (1 - \lambda)(3 - \lambda)(6 - \lambda)$

$\vec{\varphi}_1(x) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{x}, \quad \vec{\varphi}_2(x) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} e^{3x}, \quad \vec{\varphi}_3(x) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} e^{6x}$

AWP: $\vec{y}(x) = \vec{\varphi}_1(x) + \varphi_2(x) + 2\,\vec{\varphi}_3(x)$

12.23 $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \det(A - \lambda I) = (2 - \lambda)(3 - \lambda)$

$\vec{\varphi}_1(x) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2x}, \vec{\varphi}_2(x) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3x}, \vec{y}(x) = \alpha\,\vec{\varphi}_1(x) + \beta\,\vec{\varphi}_2(x)$,

$y(x) = y_1(x) = \alpha\,e^{2x} + \beta\,e^{3x}$ AWP: $y(x) = 3\,e^{2x} - 2\,e^{3x}$

12.24 $\vec{\varphi}_1(x) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \sin(x), \quad \vec{\varphi}_2(x) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cos(x)$,

$\vec{\varphi}_3(x) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2x}, \quad \vec{\varphi}_4(x) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2x}$

Differenzialgleichungen höherer Ordnung

12.25 Kein Fundamentalsystem!

12.26 Fundamentalsystem

12.27 a) $\lambda^2 + 13\lambda + 40 = (\lambda + 5)(\lambda + 8) = 0, \quad \varphi_1(t) = e^{-5t}, \quad \varphi_2(t) = e^{-8t}$

b) $\lambda^2 - 12\lambda + 36 = (\lambda - 6)^2 = 0, \quad \varphi_1(t) = e^{6t}, \quad \varphi_2(t) = t\,e^{6t}$

c) $\lambda^2 + 6\,\lambda + 34 = (\lambda + 3 - 5\,i)\ (\lambda + 3 + 5\,i) = 0$
$\varphi_1(x) = e^{-3\,x}\ \cos(5\,x)\ ,\quad \varphi_2(x) = e^{-3\,x}\ \sin(5\,x)$
d) $\lambda^2 + 16 = (\lambda + 4\,i)\ (\lambda - 4\,i) = 0\,,\quad \varphi_1(x) = \cos(4\,x)\ ,\quad \varphi_2(x) = \sin(4\,x)$

12.28 a) $\lambda^4 - 10\,\lambda^2 + 9 = (\lambda + 1)\ (\lambda - 1)\ (\lambda + 3)\ (\lambda - 3) = 0$
$\varphi_1(x) = e^{x}\,,\quad \varphi_2(x) = e^{-x}\,,\quad \varphi_3(x) = e^{3\,x}\,,\quad \varphi_4(x) = e^{-3\,x}$
b) $\lambda^3 - 2\,\lambda^2 + \lambda = \lambda\ (\lambda - 1)^2 = 0$
$\varphi_1(t) = 1\,,\quad \varphi_2(t) = e^{t}\,,\quad \varphi_3(t) = t\,e^{t}$
c) $\lambda^6 - 1 = 0 = (\lambda - 1)\ (\lambda + 1)\ \left(\lambda - \frac{1}{2} - \frac{1}{2}\sqrt{3}\,i\right)$
$\left(\lambda + \frac{1}{2} - \frac{1}{2}\sqrt{3}\,i\right)\ \left(\lambda - \frac{1}{2} + \frac{1}{2}\sqrt{3}\,i\right)\ \left(\lambda + \frac{1}{2} + \frac{1}{2}\sqrt{3}\,i\right)$
$\varphi_{1/2}(x) = e^{\pm x}$, $\varphi_{3/4}(x) = e^{\pm\frac{1}{2}\,x}\sin\left(\frac{1}{2}\sqrt{3}\,x\right)$, $\varphi_{5/6}(x) = e^{\pm\frac{1}{2}x}\cos\left(\frac{1}{2}\sqrt{3}\,x\right)$

12.29 $\lambda^2 - 3\,\lambda + 2 = (\lambda - 1)\ (\lambda - 2) = 0$
a) $y_p(x) = 3$
b) $y_p(x) = \frac{3}{4} + \frac{1}{2}\,x$
c) $y_p(x) = x\,e^{2\,x}$
d) $y_p(x) = \frac{1}{10}\ (\cos x - 3\ \sin x)$
e) $y_p(x) = 3 + 4\,x + \cos x - 3\ \sin x$
f) $y_p(x) = \left(\frac{1}{2}\,x^2 - x\right)\ e^{2\,x}$
g) $y_p(x) = -\frac{1}{2}\ (\sin x + \cos x)\ e^{x}$

12.30 a) $\lambda^2 + 16 = (\lambda - 4\,i)\ (\lambda + 4\,i) = 0\,,\quad x(t) = 3\ \cos(4\,t) + \sin(4\,t)$
b) $\lambda^2 + 2\,\lambda + 2 = (\lambda + 1 + i)\ (\lambda + 1 - i) = 0\,,\quad x(t) = 2\ \sin(t)\ e^{-t} + 2\ \cos(t)\ e^{-t}$
c) $\lambda^2 + 13\,\lambda + 40 = (\lambda + 5)\ (\lambda + 8) = 0\,,\quad x(t) = 8\,e^{-5\,t} - 5\,e^{-8\,t}$

12.31 a) $\lambda^4 - 10\,\lambda^2 + 9 = (\lambda + 1)\ (\lambda - 1)\ (\lambda + 3)\ (\lambda - 3) = 0$
$y(x) = \frac{1}{20}\ \sin(x) + C_1\,e^{-x} + C_2\,e^{x} + C_3\,e^{-3\,x} + C_4\,e^{3\,x}$
b) $\lambda^3 - 7\,\lambda - 6 = (\lambda + 1)\ (\lambda + 2)\ (\lambda - 3) = 0$
$y(x) = -e^{x} + C_1\,e^{-x} + C_2\,e^{-2\,x} + C_3\,e^{3\,x}$
c) $\lambda^3 - 2\,\lambda^2 + \lambda - 2 = (\lambda - 2)\ (\lambda + i)\ (\lambda - i) = 0$
$y(x) = -\frac{1}{5}\ (x\ \sin(x) + 2\,x\ \cos(x)) + C_1\,e^{2\,x} + C_2\ \cos(x) + C_3\ \sin(x)$
d) $\lambda^3 - 6\,\lambda^2 + 12\,\lambda - 8 = (\lambda - 2)^3 = 0$
$y(x) = x^3\,e^{2\,x} + C_1\,e^{2\,x} + C_2\,x\,e^{2\,x} + C_3\,x^2\,e^{2\,x}$

A.13 Lösungen zur Laplace-Transformation

13.1 a) $\frac{3}{s+4}$; $s > -4$
b) $\frac{4}{s^3}$; $s > 0$
c) $\frac{4\,s}{s^2+25}$; $s > 0$
d) $\frac{\pi}{s^2+\pi^2}$; $s > 0$
e) $-3\sqrt{\frac{\pi}{s}}$; $s > 0$

13.2 a) $\frac{72}{s^2} - \frac{3\sqrt{\pi}}{2\,s^{\frac{5}{2}}} + \frac{6}{s}$ b) $\frac{10-3\,s}{s^2+4}$ c) $\frac{\sqrt{\left(\frac{1}{3}\right)}}{s^{\frac{4}{3}}} + \frac{4}{s-2}$ d) nicht möglich!

13.3 a) $F(s) = \frac{A}{s}\left(1 - e^{-s\,t_0}\right) + \frac{A}{s+2}\,e^{-s\,t_0}$
b) $F(s) = \frac{A}{s}\,e^{-a\,s} - \frac{A}{s}\,e^{-b\,s}$
c) $F(s) = \frac{1}{s^2}\left(1 - e^{-3\,s}\right)$
d) $F(s) = \frac{1}{s^2+1}\left(1 + e^{-\pi\,s}\right)$

13.4 a) $5\,e^{-2\,t}$
b) $4\cos(2\,t) - \frac{3}{2}\sin(2\,t)$
c) $2 - 5\,t$
d) $\frac{t^{k-1}}{\Gamma(k)}$
e) $\frac{1}{\sqrt{\pi}}\left(8\,t^{\frac{1}{2}} - 5\,t^{-\frac{1}{2}}\right)$
f) $\frac{1}{2}\left(1 - e^{-2\,t}\right)$

13.5 a) $2\,e^t\cos(2\,t) + \frac{5}{2}\,e^t\sin(2\,t)$
b) $S(t-2)\,(t-2)$
c) $\frac{1}{6}\,S(t-5)\,(t-5)^3$
d) $\frac{1}{2}\,t^2 - \frac{1}{2}\,S(t-2)\,(t-2)^2$

13.6 a) $-\frac{4}{3}\,e^{2\,t} - \frac{1}{6}\,e^{-t} + \frac{7}{2}\,e^{3\,t}$
b) $2\,e^t - 2\cos t + \sin t$
c) $-e^{-t} - \frac{1}{2}\,t^2\,e^{2\,t} + 2\,t\,e^{2\,t} + e^{2\,t}$
d) $e^t\left(t^2 - t + 3\right)$

13.7 $y(t) = 1$

13.8 dsolve({DG, y(0)=1, (D(y))(0)=2, (D@@2)(y)(0)=3, (D@@3)(y)(0)=0}, y(t), method = laplace);

13.9 $I(t) = I_0\,e^{-\frac{R}{L}\,t}$

Anwendungen der Laplace-Transformation

13.10 $I(t) = -\frac{5}{2}\cos(5\,t) + \frac{5}{2}\sin(5\,t) + \frac{5}{2}\,e^{-5\,t}$

13.11 a) $m\,\ddot{x} = -D\,x \hookrightarrow x(t) = 5\cos\left(\sqrt{\frac{D}{m}}\,t\right)$
ω: $x(2) = 5\cos(2\,\omega) = 2.5 \Rightarrow \omega = \frac{\pi}{6} \Rightarrow x(t) = 5\cos\left(\frac{\pi}{6}\,t\right)$
b) $\dot{x}(t_0) = -5\,\frac{\pi}{6}$
c) $\ddot{x}(t_0) = 0$

13.12 $\ddot{x} + 4\,\dot{x} + 8\,x = 20\cos(2\,t)$; $x(0) = 0$, $\dot{x}(0) = 0$
$\Rightarrow X(s) = \frac{20\,s}{s^2+4} \cdot \frac{1}{s^2+4\,s+8}$
$\Rightarrow x(t) = \cos(2\,t) + 2\sin(2\,t) - e^{2\,t}\left(\cos(2\,t) + 3\sin(2\,t)\right)$

13.13 –

13.14 $X(s) = 8\,\frac{\omega}{s^2+\omega^2} \cdot \frac{1}{s^2+4} \Rightarrow x(t) = -\frac{8}{\omega^2-4}\sin(\omega t) + \frac{4}{\omega^2-4}\sin(\omega t)$
Resonanz bei $\omega = 2$

A.14

A.14 Lösungen zu Fourier-Reihen

14.1 a) $f(t) = \begin{cases} 1 & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & t = \pm\frac{\pi}{2} \\ -1 & -\pi < t < \frac{-\pi}{2}\,,\ \frac{\pi}{2} < t < \pi \end{cases}$

$f(t)$ gerade $\Rightarrow b_n = 0$ für $n = 1, 2, 3, \dots$

$a_0 = 0\,, \quad a_n = \frac{4}{\pi\, n}\left[\sin\left(n\,\frac{\pi}{2}\right)\right] = \begin{cases} \frac{4}{n\,\pi} & n = 1, 5, 9, \dots \\ -\frac{4}{n\,\pi} & n = 3, 7, 11, \dots \\ 0 & n \ \text{gerade} \end{cases}$

$$\Rightarrow f(t) = \frac{4}{\pi}\left\{\cos(t) - \frac{1}{3}\cos(3\,t) + \frac{1}{5}\cos(5\,t) - \frac{1}{7}\cos(7\,t) \pm \dots\right\} = \frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^n\,\frac{1}{2n+1}\cos((2\,n+1)\,t)$$

b) $f(t) = \left(1 - \frac{x}{2\pi}\right)$ für $0 < x < 2\pi$

$a_0 = \frac{1}{2}\,, \quad a_n = 0\,, \quad b_n = \frac{1}{\pi\,n} \quad \Rightarrow f(t) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\,t)$

Der Fourierwert an der Stelle $t = 0$ ist $f(0) = \frac{1}{2}$.

14.2 a) f gerade, $b_n = 0\,,\ f(t) = \frac{2}{T}\left(1 - e^{-\frac{T}{2}}\right) + 4\,T\sum_{n=1}^{\infty}\frac{1-(-1)^n\,e^{-\frac{T}{2}}}{T^2+4\,n^2\,\pi^2}\cos\left(n\,\frac{2\pi}{T}\,t\right)$

b) $f(t) = \frac{3}{4}\,h - \frac{2\,h}{\pi^2}\sum_{\substack{n=1\\ n\ \text{ungerade}}}^{\infty}\frac{1}{n^2}\cos\left(n\,\frac{2\pi}{T}\,t\right) - \frac{h}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\,\frac{2\pi}{T}\,t\right)$

14.3 a) $c_n = \frac{1}{n\,\pi}\,2\,\frac{1}{2i}\left[e^{i\,n\,\frac{\pi}{2}} - e^{-i\,n\,\frac{\pi}{2}}\right] = \frac{2}{n\,\pi}\sin\left(n\,\frac{\pi}{2}\right)$

$= \begin{cases} \frac{2}{n\,\pi} & \text{für} \ \ n = 1, 5, 9, \dots \\ -\frac{2}{n\,\pi} & \text{für} \ \ n = 3, 7, 11, \dots \\ 0 & \text{für} \ \ n \ \text{gerade} \end{cases}$

b) $c_n = \frac{1}{2\pi}\,\frac{1}{i\,n} = -i\,\frac{1}{2\pi\,n}\,, \quad c_0 = \frac{1}{2} \quad \Rightarrow f(t) = \frac{-i}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n}\,e^{i\,n\,t} + \frac{1}{2}$

14.4 Nach den Moivreschen Formeln gilt

$\sin^3(t) = \frac{1}{4}\,(3\cdot\sin(t) - \sin(3\,t)) = \frac{3}{4}\sin(t) - \frac{1}{4}\sin(3t)$.

14.5 a) $f(t) = \frac{1}{3}\,T^2 + \sum_{n=1}^{\infty}\left(\frac{T^2}{n^2\,\pi^2}\cos\left(n\,\frac{2\pi}{T}\,t\right) - \frac{T^2}{n\,\pi}\sin\left(n\,\frac{2\pi}{T}\,t\right)\right)$

b) $f(t) = \frac{4}{3}\,\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos(n\,t) - \frac{4\pi}{n}\sin(n\,t)\right)$

c) $f(2\pi) = \frac{(2\pi)^2+0}{2} = 2\pi^2 = \frac{4}{3}\,\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

14.6 $a_0 = \frac{u_0}{2}\,, \quad a_n = 0\,, \quad b_n = -\frac{u_0}{\pi}\,\frac{1}{n} \Rightarrow u(t) = \frac{u_0}{2} - \frac{u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\,\frac{2\pi}{T}\,t\right)$

Gleichspannungsanteil $\frac{u_0}{2}$

Grundschwingung mit Amplitude $\frac{u_0}{\pi}$ und Frequenz $\omega_0 = \frac{2\pi}{T}$

Sinusförmige Oberschwingungen mit Amplitude $\frac{u_0}{2\pi}\,,\ \frac{u_0}{3\pi}\,,\ \frac{u_0}{4\pi}\,, \dots$ bei den Frequenzen $2\,\omega_0\,,\ 3\,\omega_0\,,\ 4\,\omega_0, \dots$.

14.7 a) $T = 4\,, \quad \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$. Funktion ungerade $\Rightarrow a_0 = 0\,, \quad a_n = 0 \quad n \in \mathbb{N}$.

$b_n = \frac{16}{n\,\pi}\,(1 - \cos(n\,\pi)) = \begin{cases} 0 & n \ \text{gerade} \\ \frac{32}{n\,\pi} & n \ \text{ungerade} \end{cases}$

$\Rightarrow f(x) = \sum_{\substack{n=1\\ n\ \text{ungerade}}}^{\infty}\frac{32}{n\,\pi}\sin\left(n\,\frac{\pi}{2}\,x\right)$

b) $T = 8\,, \quad \omega_0 = \frac{2\pi}{T} = \frac{\pi}{4}$. Funktion gerade $\Rightarrow b_n = 0 \quad n \in \mathbb{N}$.

$a_0 = 2\,, \quad a_n = \frac{8}{n^2\,\pi^2}\,(\cos(n\,\pi) - 1) = \begin{cases} 0 & n \ \text{gerade} \\ -\frac{16}{n\,\pi} & n \ \text{ungerade} \end{cases}$

$\Rightarrow f(x) = 2 - \sum_{\substack{n=1\\ n\ \text{ungerade}}}^{\infty}\frac{16}{n^2\,\pi^2}\cos\left(n\,\frac{\pi}{4}\,x\right)$.

14.8 $\frac{3}{4} - \frac{2}{\pi^2}\sum_{\substack{n=1\\ n\ \text{ungerade}}}^{\infty}\frac{1}{n^2}\cos\left(n\,\frac{2\pi}{T}\,t\right) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\,\frac{2\pi}{T}\,t\right)$.

14.9 a) $20 - \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(n \frac{\pi}{5} t\right)$
b) $\frac{3}{2} - \frac{12}{\pi^2} \sum_{\substack{n=1 \\ n \text{ ungerade}}}^{\infty} \frac{1}{n^2} \cos\left(n \frac{\pi}{3} t\right) - \frac{6}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \sin\left(n \frac{\pi}{3} t\right).$

A.15 Lösungen zur Fourier-Transformation

A.15

15.1 a) $\mathcal{F}(f_1)(\omega) = 2A \frac{\sin\left(\omega \frac{T}{2}\right)}{\omega} = A\,T \frac{\sin\left(\omega \frac{T}{2}\right)}{\omega \frac{T}{2}} = A\,T\, si\left(\omega \frac{T}{2}\right)$
b) Für $A = \frac{1}{T}$ ergibt sich $\mathcal{F}(f_1)(\omega) = si\left(\omega \frac{T}{2}\right)$.
c) $\mathcal{F}(f_2)(\omega) = 2A\, e^{-i\,\omega\, t_0} \frac{\sin\left(\omega \frac{T}{2}\right)}{\omega} = e^{-i\,\omega\, t_0} F(f_1)(\omega)$

15.2 $\mathcal{F}(f)(\omega) = A\,T \frac{\sin^2\left(\omega \frac{T}{2}\right)}{\left(\omega \frac{T}{2}\right)^2} = A\,T\, si^2\left(\omega \frac{T}{2}\right) = \frac{2A}{\omega^2 T}(\cos(\omega T) - 1)$

15.3 a) f ungerade $\hookrightarrow \mathcal{F}(f)(\omega) = -2i \int_0^{\infty} f(t) \sin(\omega t)\, dt = -i \frac{2\,\omega}{\alpha^2 + \omega^2}$
b) $\mathcal{F}(f)(\omega) = \frac{\sin(\omega T)}{\omega\left(1 - \left(\frac{\omega T}{\pi}\right)^2\right)}$

15.4 > fourier (f(t), t, w);

15.5 Die Funktion setzt sich zusammen aus der Funktion f_1, aus Aufgabe 15.1a) mit der Breite $2T$ und der Dreiecksfunktion aus Aufgabe 15.2 . Mit der Skalierungseigenschaft und der Linearität der Fourier-Transformation erhält man
$F(f)(\omega) = 2A\,T \frac{\sin(\omega T)}{\omega T} + A\,T \frac{\sin^2\left(\omega \frac{T}{2}\right)}{\left(\omega \frac{T}{2}\right)^2} = 2A\,T\, si(\omega T) + A\,T\, si^2\left(\omega \frac{T}{2}\right).$

15.9 a) $\mathcal{F}(\delta(t)) = 1$
b) $\mathcal{F}(\delta(t - t_0)) = e^{-i\omega t_0}$
c) $\mathcal{F}\left(\frac{i}{2}\left(\delta(t + t_0) - \delta(t - t_0)\right)\right) = -\sin(\omega\, t_0)$
d) $\mathcal{F}(\sin(\omega_0 t)) = \frac{\pi}{i}\left(\delta(\omega - \omega_0) - \delta(\omega - \omega_0)\right)$

15.10 a) $\mathcal{F}\left(e^{i\,a\,t}\right)(\omega) = \mathcal{F}\left(e^{i\,a\,t} \cdot 1\right)(\omega) \underset{\substack{\uparrow \\ \text{Verschiebungssatz}}}{=} \mathcal{F}(1)(\omega - a) = 2\pi\, \delta(\omega - a)$
b) $\delta(t - a) * f(t) = \int_{-\infty}^{\infty} \delta\underbrace{(t - a - \tau)}_{\substack{=0 \\ \tau = t - a}} f(\tau)\, d\tau = f(t - a)$

15.11 $rect\left(\frac{2t}{T}\right) * rect\left(\frac{2t}{T}\right) = tri\left(\frac{t}{T}\right) = \begin{cases} 1 - |t| & |t| \le T \\ 0 & |t| > T \end{cases}$

15.12 a) $\frac{1}{i\omega}\left(1 - e^{i\omega T}\right)^2 = \frac{-4}{i\omega} e^{i\omega T} \sin^2\left(\frac{\omega T}{2}\right)$
b) $-2i\left[-\frac{T}{\omega} \cos(\omega t) + \frac{\sin(\omega T)}{\omega^2}\right]$
c) $\frac{2}{\omega}\left[\sin(\omega T_2) - \sin(\omega T_1)\right]$

15.13 $t < 0$: $(f * h)(t) = 0$ $\quad 0 \le t \le T$: $(f * h)(t) = t - 1 + e^{-t}$
$T < t$: $(f * h)(t) = T - e^{-t}\left(e^T - 1\right)$

A.16

A.16 Lösungen zu Partielle Differenzialgleichungen

16.1 $u\,(x,\,t) = \cos(\omega t)\,\sin(k\,x)\,;\; u_{xx}\,(x,\,t) = -k^2\,\cos(\omega t)\,\sin(k\,x)\,;$
$u_{tt}\,(x,\,t) = -\omega^2\,\cos(\omega t)\,\sin(k\,x)$
$u_{tt} - c^2\,u_{xx} = \left(-\omega^2 + c^2\,k^2\right)\,\cos(\omega t)\,\sin(k\,x) = 0 \quad \text{für } \omega = c\,k$
$u\,(x = 0,\,t) = \cos(\omega t)\,\sin(0) = 0;\; u\,(x = L,\,t) = \cos(\omega t)\,\sin\left(n\,\frac{\pi}{L}\right)\,L = 0 \quad \text{für } k = n\,\frac{\pi}{L}$

16.2 $u\,(x,\,t) = e^{-\omega t}\,\sin(k\,x)\,;\; u_{xx}\,(x,\,t) = -k^2\,e^{-\omega t}\,\sin(k\,x)\,;$
$u_t\,(x,\,t) = -\omega\,e^{-\omega t}\,\sin(k\,x)$
$u_t - D\,u_{xx} = \left(-\omega + D\,k^2\right)\,e - \omega t\,\sin(k\,x) = 0 \quad \text{für } \omega = D\,k^2$

16.3 a) $f\,(x,\,y) = \frac{1}{2}\,\ln\left(x^2 + y^2\right)\,;\; f_{xx}\,(x,\,y) = -\frac{x^2-y^2}{\left(x^2+y^2\right)^2};\; f_{yy}\,(x,\,y) = \frac{x^2-y^2}{\left(x^2+y^2\right)^2};$
$f_{xx} + f_{yy} = 0$
b) $g\,(x,\,y,\,z) = \left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}}$
$g_{xx}\,(x,\,y,\,z) = 3\,\left(x^2 + y^2 + z^2\right)^{-\frac{5}{2}}\,x^2 - \left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}};\; g_{yy}\,,\, g_{zz}$ analog
$g_{xx} + g_{yy} + g_{zz} = 3\,\left(x^2 + y^2 + z^2\right)^{-\frac{5}{2}}\,\left[x^2 + y^2 + z^2\right] - 3\,\left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}} = 0$

16.4 a) $u\,(t) = f_1\,(x + c\,t) + f_2\,(x - c\,t);\; u_{tt} = c^2\,\left(f_1''\,(x + c\,t) + f_2''\,(x - c\,t)\right);$
$u_{xx} = f_1''\,(x + c\,t) + f_2''\,(x - c\,t) \quad ; \quad u_{tt} - c^2\,u_{xx} = \ldots = 0$
b) $u\,(x,\,t) = \frac{1}{2}\,\left(u_0\,(x + c\,t) + u_0\,(x - c\,t)\right) + \frac{1}{2}\,c\,\int_{x-ct}^{x+ct} v_0\,(\zeta)\,d\zeta$
c) $u\,(x,\,t) = \frac{1}{2}\,\left(\sin\left(k\,(x + c\,t)\right) + \sin\left(k\,(x - c\,t)\right)\right) = \sin(k\,x) \cdot \sin(k\,c\,t) = \sin(k\,x) \cdot \sin(\omega t) \quad \text{mit } \omega = c\,k$

16.5 $\partial_x\,\frac{1}{R} = -\frac{x-a}{R^3} \quad ; \quad \partial_x^2\,\frac{1}{R} = -\frac{1}{R^3} + \frac{3\,(x-a)^2}{R^5} \quad ; \ldots$

16.7 $k_n = \kappa\,\left(\frac{n\,\pi}{L}\right)^2 \quad ; \quad u\,(x,\,t) = \sum_{n=1}^{\infty} c_n\,\sin\left(\frac{n\,\pi}{L}\,x\right)\,e^{-\left(\frac{n\,\pi}{L}\right)^2\,t}$

16.8 $u\,(x,\,t) = T\,(t) \cdot X\,(x) \Rightarrow \frac{T'}{T} = \frac{X''}{X} = k^2$
$\Rightarrow u\,(x,\,t) = A\,e^{k^2\,t}\,\left(B\,\sin(k\,x) + C\,\cos(k\,x)\right)$

16.9 a) $u\,(x,\,y) = X\,(x) \cdot Y\,(y) \Rightarrow \frac{X''}{X} = \frac{Y''}{Y} = -k^2$
$\Rightarrow u\,(x,\,y) = \left[A\,\cos(k\,x) + B\,\sin(k\,x)\right]\,\left[C\,\cos(k\,y) + D\,\sin(k\,y)\right]$
b) $X\,(0) = 0 \Rightarrow A = 0; \quad Y\,(0) = 0 \Rightarrow C = 0; \quad Y\,(L) = 0 \Rightarrow k_n = n\,\frac{\pi}{L}$
$\Rightarrow u\,(x,\,y) = \sum_{n=1}^{\infty} c_n\,\sin\left(n\,\frac{\pi}{L}\,x\right)\,\sin\left(n\,\frac{\pi}{L}\,y\right)$

16.10 a) $k = \pm\,i\,\sqrt{n^2 + m^2}\,\frac{2\pi}{L}$
b) $u_1 = \sin\left(n\,\frac{2\pi}{L}\,x\right)\,\sin\left(m\,\frac{2\pi}{L}\,y\right)\,\sin\left(\sqrt{n^2 + m^2}\,\frac{2\pi}{L}\,t\right)$
$u_2 = \sin\left(n\,\frac{2\pi}{L}\,x\right)\,\sin\left(m\,\frac{2\pi}{L}\,y\right)\cos\left(\sqrt{n^2 + m^2}\,\frac{2\pi}{L}\,t\right)$

16.11 $k = \left(n^2 + m^2\right)\,\frac{4\pi^2}{L^2}$

16.12 a) $u_{xx} = -k^2\,\sin(k\,x)\,\left(e^{k\,y} + e^{-k\,y}\right) \qquad u_{yy} = k^2\,\sin(k\,x)\,\left(e^{k\,y} + e^{-k\,y}\right)$
b) $k = n\,\frac{\pi}{L}$

16.13 $u\,(x,\,t) = 1 - \frac{x}{\pi} + \frac{2}{\pi}\,\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\,e^{-n^2\,t}\,\sin(n\,x)$